Friday, December 4, 2015

p. 644: 5, 8, 9, 13, 14, 15, 20, 22, 24

Problem 5

Problem. Find a first-degree polynomial function P_1 whose value and slope agree with the value and slope of $f(x) = \dfrac{\sqrt{x}}{4}$ at $x = 4$.

Solution.

$$\begin{aligned} f(x) &= \frac{\sqrt{x}}{4}, \\ f(4) &= 1, \\ f'(x) &= \frac{1}{8}x^{-1/2}, \\ f'(4) &= \frac{1}{16}. \end{aligned}$$

So, $P_1(x) = 1 + \dfrac{(x-4)}{16}$.

Problem 8

Problem. Find a first-degree polynomial function P_1 whose value and slope agree with the value and slope of $f(x) = \tan x$ at $x = \frac{\pi}{4}$.

Solution.

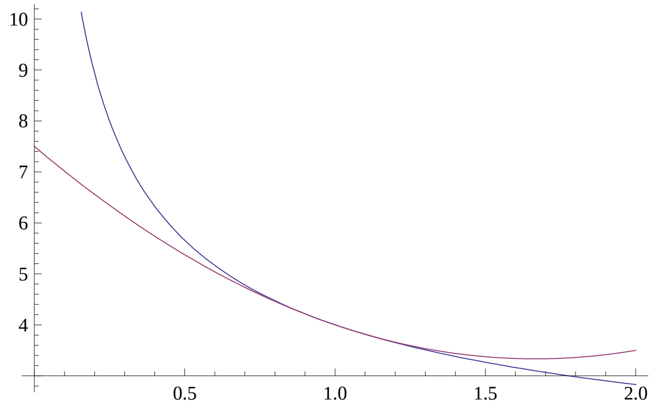
$$\begin{aligned} f(x) &= \tan x, \\ f(\tfrac{\pi}{4}) &= 1, \\ f'(x) &= \sec^2 x, \\ f'(\tfrac{\pi}{4}) &= 2. \end{aligned}$$

So, $P_1(x) = 1 + 2(x - \frac{\pi}{4})$.

Problem 9

Problem. Use a graphing utility to graph $f(x) = \dfrac{4}{\sqrt{x}}$ and its second-degree polynomial approximation $P_2(x) = 4 - 2(x-1) + \frac{3}{2}(x-1)^2$ at $c = 1$. Complete the table comparing values of f and P_2.

Solution. The graphs:



The table:

x	0	0.8	0.9	1	1.1	1.2	2
$f(x)$	∞	4.4721	4.2163	4	3.8138	3.6514	2.8284
$P_2(x)$	7.5	4.46	4.215	4	3.815	3.66	3.5

Problem 13

Problem. Find the 4th Maclaurin polynomial for the function $f(x) = e^{4x}$.

Solution. The table of coefficients:

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$\frac{f^{(n)}(0)}{n!}$
0	e^{4x}	1	1
1	$4e^{4x}$	4	4
2	4^2e^{4x}	4^2	$\frac{4^2}{2!}$
3	4^3e^{4x}	4^3	$\frac{4^3}{3!}$
4	4^4e^{4x}	4^4	$\frac{4^4}{4!}$

$$\begin{aligned} P_4(x) &= 1 + 4x + \frac{4^2x^2}{2!} + \frac{4^3x^3}{3!} + \frac{4^4x^4}{4!} \\ &= 1 + 4x + 8x^2 + \tfrac{32}{3}x^3 + \tfrac{32}{3}x^4. \end{aligned}$$

Problem 14

Problem. Find the 5th Maclaurin polynomial for the function $f(x) = e^{-x}$.

Solution. The table of coefficients:

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$\frac{f^{(n)}(0)}{n!}$
0	e^{-x}	1	1
1	$-e^{-x}$	-1	-1
2	e^{-x}	1	$\frac{1}{2!}$
3	$-e^{-x}$	-1	$-\frac{1}{3!}$
4	e^{-x}	1	$\frac{1}{4!}$
5	$-e^{-x}$	-1	$-\frac{1}{5!}$

$$\begin{aligned} P_4(x) &= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} \\ &= 1 - x + \tfrac{1}{2}x^2 - \tfrac{1}{6}x^3 + \tfrac{1}{24}x^4 - \tfrac{1}{120}x^5. \end{aligned}$$

Problem 15

Problem. Find the 4th Maclaurin polynomial for the function $f(x) = e^{-x/2}$.

Solution. The table of coefficients:

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$\frac{f^{(n)}(0)}{n!}$
0	$e^{-x/2}$	1	1
1	$-\frac{e^{-x/2}}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
2	$\frac{e^{-x/2}}{2^2}$	$\frac{1}{2^2}$	$\frac{1}{2^2 \cdot 2!}$
3	$-\frac{e^{-x/2}}{2^3}$	$-\frac{1}{2^3}$	$-\frac{1}{2^3 \cdot 3!}$
4	$\frac{e^{-x/2}}{2^4}$	$\frac{1}{2^4}$	$\frac{1}{2^4 \cdot 4!}$

$$\begin{aligned} P_4(x) &= 1 - \frac{x}{2} + \frac{x^2}{2^2 \cdot 2!} - \frac{x^3}{2^3 \cdot 3!} + \frac{x^4}{2^4 \cdot 4!} \\ &= 1 - \tfrac{1}{2}x + \tfrac{1}{8}x^2 - \tfrac{1}{48}x^3 + \tfrac{1}{384}x^4. \end{aligned}$$

Problem 20

Problem. Find the 4th Maclaurin polynomial for the function $f(x) = x^2 e^{-x}$.

Solution. Now things start to get a little messy. We need to compute the first 4 derivatives of x^2e^{-x}.

$$\begin{aligned} f'(x) &= 2xe^{-x} - x^2e^{-x} \\ &= (2x - x^2)e^{-x}, \\ f''(x) &= (2 - 2x)e^{-x} - (2x - x^2)e^{-x} \\ &= (2 - 4x + x^2)e^{-x}, \\ f'''(x) &= (-4 + 2x)e^{-x} - (2 - 4x + x^2)e^{-x} \\ &= (-6 + 6x - x^2)e^{-x}, \\ f^{(4)}(x) &= (6 - 2x)e^{-x} - (-6 + 6x - x^2)e^{-x} \\ &= (12 - 8x + x^2)e^{-x}. \end{aligned}$$

The table of coefficients:

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$\frac{f^{(n)}(0)}{n!}$
0	x^2e^{-x}	0	0
1	$(2x - x^2)e^{-x}$	0	0
2	$(2 - 4x + x^2)e^{-x}$	2	$\frac{2}{2!}$
3	$(-6 + 6x - x^2)e^{-x}$	-6	$-\frac{6}{3!}$
4	$(12 - 8x + x^2)e^{-x}$	12	$\frac{12}{4!}$

$$\begin{aligned} P_4(x) &= x^2 - x^3 + \frac{x^4}{2} \\ &= x^2 - x^3 + \tfrac{1}{2}x^4. \end{aligned}$$

There is a quick way to work this problem. We already know that the second-degree Taylor Polynomial for e^{-x} is $1 - x + \frac{1}{2}x^2$. We could simply multiply it termwise by x^2 to get the fourth-degree Taylor polynomial for x^2e^{-x}.

Problem 22

Problem. Find the 4th Maclaurin polynomial for the function $f(x) = \dfrac{x}{x+1}$.

Solution. We need to compute the first 4 derivatives of $\dfrac{x}{x+1}$.

$$\begin{aligned}
f'(x) &= \frac{(x+1)\cdot 1 - 1\cdot x}{(x+1)^2} \\
&= \frac{1}{(x+1)^2}, \\
f''(x) &= -\frac{2!}{(x+1)^3}, \\
f'''(x) &= \frac{3!}{(x+1)^4}, \\
f^{(4)}(x) &= -\frac{4!}{(x+1)^4}.
\end{aligned}$$

The table of coefficients:

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$\frac{f^{(n)}(0)}{n!}$
0	$\frac{x}{x+1}$	0	0
1	$\frac{1}{(x+1)^2}$	1	1
2	$-\frac{2!}{(x+1)^3}$	$2!$	$-\frac{2!}{2!} = -1$
3	$\frac{3!}{(x+1)^4}$	$3!$	$-\frac{3!}{3!} = 1$
4	$-\frac{4!}{(x+1)^4}$	$-4!$	$\frac{-4!}{4!} = -1$

$$P_4(x) = x - x^2 + x^3 - x^4.$$

We could work this problem much faster if we noted that $f(x) = x \cdot \dfrac{1}{x+1}$ and that $\dfrac{1}{x+1}$ can be expanded as a geometric series:

$$\begin{aligned}
\frac{1}{1+x} &= \frac{1}{1-(-x)} \\
&= 1 - x + x^2 - x^3 + \cdots .
\end{aligned}$$

Then multiply by x and use the terms up to x^4 to get $x - x^2 + x^3 - x^4$.

Problem 24

Problem. Find the 3rd Maclaurin polynomial for the function $f(x) = \tan x$.

Solution. We need to compute the first 3 derivatives of $\tan x$.

$$\begin{aligned}
f'(x) &= \sec^2 x, \\
f''(x) &= 2\sec x \cdot \sec x \tan x \\
&= 2\sec^2 x \tan x, \\
f'''(x) &= (4\sec x \cdot \sec x \tan x)(\tan x) + (2\sec^2 x)(\sec^2 x) \\
&= 4\sec^2 x \tan^2 x + 2\sec^4 x.
\end{aligned}$$

The table of coefficients:

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$\frac{f^{(n)}(0)}{n!}$
0	$\tan x$	0	0
1	$\sec^2 x$	1	1
2	$2\sec^2 x \tan x$	0	0
3	$4\sec^2 x \tan^2 x + 2\sec^4 x$	2	$\frac{2}{3!} = \frac{1}{3}$

$$\begin{aligned}
P_3(x) &= x + \frac{x}{3} \\
&= x + \tfrac{1}{3}x.
\end{aligned}$$